**Unit 8: Investigation 5 (4 Days)**

**Applications with** *n* × *n* **Matrices**

**Common Core Stare Standards**

N-VM 6: Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.

N-VM 8*:* Add, subtract, and multiply matrices of appropriate dimensions*.*

N-VM 9: Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.

**Overview**

In this investigation we find determinants of a *n* × *n* (for *n* = 3) matrices using cofactor expansions. We use the matrix of cofactors to find the inverse of an *n* × *n* matrix. The inverse matrix will then be used to solve systems of equations as in investigation 8.4.2, but in this investigation with 3 × 3 matrices. Investgation 8.5.5 examines an area application of determinants. Investigation 8.5.6 enhances previous knowledge of matrices as transformations including transformations of points, lines, and flat surfaces i.e. *n*-gons in two and three-space.

**Assessment Activities**

**Evidence of Success: What Will Students Be Able to Do?**

* Students will find the determinant of a 3 × 3 matrix using cofactor expansions. Students will recognize that this procedure produces identical results when done across rows or down columns.
* Students will rearrange a system of three linear equations in three variables into standard form and then create a coefficient matrix. Students will determine the inverse of a coefficient matrix using the matrix of cofactors.
* Students will verify their work using technology. Students will use the inverse of the 3 × 3 matrix to find the solution for linear systems of three equations in three variables.
* Students will use technology to solve more complex contextual problems involving linear equations in three variables.
* Students will map points in three-space by multiplying vectors by matrices and transform geometric figures using a transformation matrix.
* Students will work with 3 × 3 matrices as transformations of the plane.
* Students will interpret the absolute value of a special determinant in terms of area.

**Assessment Strategies: How will they show what they know?**

* **Exit Slip 8.5.1** assesses student’s ability to find the cofactors of a three by three matrix. The cofactors will be used in Activity 8.5.1.
* **Exit Slip 8.5.2** assesses student’s ability to find the determinant of a more complicated three by three matrix using cofactor expansion across different rows and columns.
* **Exit Slip 8.5.3** assesses student’s ability to create modifications in the original recycling scenario, thus producing new systems of equations to solve. The student then produces the corresponding matrix equations and uses technology to find the inverse of the coefficient matrix and finally solves the new system using matrix algebra.
* **Journal Prompt 1** Since the inverse of a matrix returns a point to where it was, what do you think happens when we move points by multiplying them be singular matrices? (A matrix without an inverse).
* **Activity 8.5.1 Solving A Recycling Matrix Equation** has students see a contextual problem represented by a system of three equations in three variables. They model the solution based on their experience with 2X2 matrix equations in an earlier activity and solve the matrix equation using technology to find the inverse matrix. It then introduces students to minors and cofactors of a matrix.
* **Activity 8.5.2** **Cofactors, Determinants and Inverses** has students find the determinant of a matrix using all 6 cofactor expansions. It also presents the method to compute the inverse of a 3 X 3 matrix, when it exists,using cofactors.
* **Activity 8.5.3 Solving a 3×3 Contextual Application using Cofactors and Inverses** has students solve the problem from activity 8.5.1 but without the technology used earlier to find the inverse matrix. Instead students use their new skills and understandings and find the inverse matrix using determinants and cofactors.
* **Activity 8.5.4 Using Technology to Solve Matrix Equations** has studentssolve the system from Activity 8.5.3 using technology but after first changing value of an output or first making changes in the coefficient matrix changing the context slightly.
* **Activity 8.5.5 Using Matrices to Find Triangular Areas** introduces students to an interesting application of determinants, namely finding the area of a triangle if you have the coordinates of its vertices.
* **Activity 8.5.6** **Using Matrices as Transformations of Points, Lines and Various *n*-gons** has students explore the effect of multiplying a vector by a matrix to give a new vector so matrix multiplication effectively moves points to new locations. They also discover that when a matrix has an inverse, multiplying the new vector by the inverse yields the original vector. Students extend the context and see that points, lines, triangles can be moved in space by multiplying by an appropriate vector or matrix.

**Launch Notes**

Begin this investigation with a review of a linear system in two variables and the steps in the procedure to solve the system using the inverse matrix. Since we know how to find the inverse of a two by two matrix, choose any convenient context.

Here is one: Two basketball teams are going into the third period with Team A leading Team B 70 to 64. Team A has made as many three point shots as team B has made two point shots. Team B has made two fewer threes than Team A has made two point shots. They both have made ten free throws.

How many of each did each team make? The students work through this review example in Activity 8.5.1.

So we get this matrix equation

Use the formula for the inverse matrix:

To find the inverse matrix:

Multiply the inverse matrix byto find the solution:

So Team A scored 12 threes and 12 twos for a total score of 36 + 24 + 10 = 70 while

Team B scored 10 threes and 12 twos for a total score of 30 + 24 + 10 = 64.

What’s important in this review is creating equations from a context and using the matrix equation procedures to find the solution. Stress that any linear system can be put into matrix format and solved using the inverse matrix provided the inverse matrix exists.

**Teaching Strategies**

In **Activity 8.5.1 Solving A Recycling Matrix Equation**, students see a contextual problem represented by a system of three equations in three variables. Their ultimate challenge is to write a matrix equation, then find its determinant using a cofactor expansion and ultimately the inverse matrix. After including the launch example, students return to the recycling context in activity 8.1.4 and 8.3.1. Students are given the context that their school’s reduce/reuse/recycle initiative not only includes the textile collection found in the earlier activities, but is encouraging families to improve recycling efforts with bottles, cans, and newsprint. This naturally leads to assembling and analyzing data which is often linear and often involves more than two variables. Students set up the matrix equation and solve it using the inverse matrix. But technology is used to find the inverse matrix and the question raised is can we find the inverse matrix without technology. This activity can be done individually or in pairs.Minors and cofactors are defined. **Exit Slip 8.5.1** should be distributed. Each student should be asked to find 3 cofactors along one row or one column. The cofactor entries they determine will be used in Activity 8.5.2.

**Activity 8.5.2** **Cofactors, Determinants and Inverses** is a good one for a group activity. Students’ results from their exit slips can be used to find the determinant of the matrix using different cofactor expansions. Students are then guided through the process of finding the inverse for a 3 by 3 matrix using cofactors and the determinant of the matrix.

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| **Group Activity 8.5.2** Cofactors, Determinants and Inverses  Groups of two to four students. Students will continue with context from Activity 8.5.1. The goal is to find the inverse of a matrix. There are three steps. Starting with this matrix:, create a new matrix called the matrix of cofactors. Each entry in the new determinant is the cofactor for the entry in the original matrix. Students can refer to their results from exit slip 8.5.1 for the cofactors.  Step 2: Move the entries across the diagonal:  Step 3: Multiply by which gives:  After all this work, they should check their results. If the matrices are inverses, they should expect to get the identity matrix of order 3. To ease the process, students can use technology to see what you get when you multiply:  Students should also check the product:  Since matrix multiplication is not in general commutative, this is an important result. If A and B are square matrices and AB = I then BA = I |

Having learned the procedure for finding the inverse of a 3×3 matrix students can now solve any system of three linear equations with three variables with challenging coefficients. This is done exactly the same way as with a two by two matrix and following the procedural steps in activity 8.5.2. **Exit slip 8.5.2** can be used and the answers will be used in activity 8.5.3.

**In activity 8.5.3 Solving a 3×3 Contextual Application using Cofactors and Inverses** hasstudents complete the solution of the problem begun in activity 8.5.1 and after completing it will hopefully appreciate the convenience of technology when finding an inverse of a matrix.

**Differentiated Instruction (For a struggling learner)** Activities 8.5.3 and 8.5.4 are a challenge because of the numerous steps and precision required. A single error causes the entire solution to be incorrect and they can be difficult to trace. For the struggling student, you can make up some very easy matrices to work with. The point is to have the procedural steps done correctly and in the correct order at least once so students can appreciate the convenience of being able to use technology. Some students or classes can just do one problem with teacher guidance and then use technology for the remainder of the unit. It is not necessary for them to be assessed on the by hand procedure. It is just important that they know a procedure does exist so that the inverse key is just not a black box.

**Activity 8.5.4 Using Technology to Solve Matrix Equations** has students revisit the problem from Activity 8.5.3 and now solve using it technology rather than solving by hand using cofactors and determinants. It also asks students to change some of the quantities from the original problem and use technology to find the solutions to the new systems**. Exit Slip** **8.5.3** can now be used.

**Differentiated Instruction (For the advanced learner)** Activities 8.5.3 and 8.5.4 can be extended to a 4×4 matrix. Since students already know how to find the determinant of a 3×3 matrix, they should be able to find the cofactor matrix for a 4×4 matrix. In particular, see if they can devise a context that results in four linear equations with four variables and to set up the matrix equation that would need to be solved. They can go as far by hand as they are able, and of course they have technology to solve the system once they have the matrix equation set up.

**Activity 8.5.5 Using Matrices to find Triangular Areas** introduces students to an interesting application of determinants, namely finding the area of a triangle if you have the coordinates of its vertices. For students needing some challenge, they could research the derivation of the formula, obtaining an excellent algebra workout and present their research to the class.

**Activity 8.5.6** **Using Matrices as Transformations of Points, Lines and Various *n*-gons** has students extend the work they did Investigation 8.4 where they saw how matrices can be used to represent transformations of points in two-space. Multiplying a vector by a matrix gives a new vector, so matrix multiplication effectively moves points to new locations. This can be undone by multiplying by the inverse matrix. In three-space, the point in Cartesian coordinates can be considered a vector where the vector is drawn from the origin to the point. If we multiply this vector by the matrix , we see the result essentially stretches the fabric of three-space so that every point moves twice as far away from the origin.

What matrix would reverse this, moving the point back to ? Matrices can move one point, several points, a line of points all at the same time. A triangle can be transformed with a matrix by applying matrix multiplication to a matrix containing the three vertices of a triangle. This is done in this activity using a two by two matrix for simplicity.

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| **Journal Prompt 1**  Since the inverse of a matrix returns a point to where it was, what do you think happens when we move points by multiplying them by singular matrices? (A matrix without an inverse). Possible answer: The transformation cannot be reversed. |

**Closure Notes**

The investigation develops some mathematical depth to students understanding of matrices as useful mathematical tools. Central to this is the idea of the inverse of a matrix and the sometimes daunting procedural steps to find the inverse if it exists. The effort is worth the work and the result is efficiently solving systems of linear equations in three or more variables, especially when it is desired to be able to change the value of the outputs without redoing the entire process. The final activity employs matrices as transformation of points, lines, triangles or equivalent objects in any dimensional space and for invertible matrices, makes the process reversible. Time permitting and student interest, a discussion of 4-space could be encouraged and students could just begin the process of determining the determinant of a 4X4 matrix and finding its inverse when it exists.

**Vocabulary**

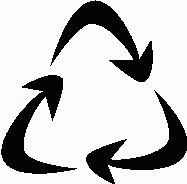
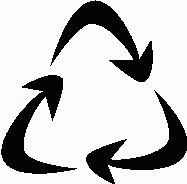
Cofactor

Determinant

Minor

**Resources and Materials**

**All activity sheets should be completed**

* Activity 8.5.1Solving the Recycling Matrix Equation
* Reduce    Reuse     Recycle: The value of recycled materials

<http://www.ct.gov/deep/cwp/view.asp?a=2714&q=324884&deepNav_GID=1645http://www.ct.gov/deep/cwp/view.asp?a=2714&q=324884&deepNav_GID=1645>

* Activity 8.5.2Cofactors, determinants and inverses
* Activity 8.5.3 Solving a 3×3 Contextual Application using Cofactors and Inverses
* Activity 8.5.4Using Technology to Solve System of Equations with 3×3 Matrices

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* Activity 8.5.5 Using Matrices to Find Triangular Areas
* Activity 8.5.6 Using Matrices as Transformations of Points, Lines and Various *n*-gons
* Bulletin board for key concepts
* Graphing Calculators
* Student Journals
* Projector
* Computers
* Rulers