**Activity 6.2.1 The Wrapping Function**

**Part 1**

Investigation 6.1 was about arcs on the unit circle. You made your own special tape measure and wrapped it around the circle to find the length of an arc on a circle of radius 1. We placed the circle on the coordinate plane and found the measure of some of the central angles of the circle. When the circle has a radius of 1 unit, we found that the radian measure of a central angle equals the length of the arc on the unit circle subtended by the angle. The question at hand is how to determine the coordinates of the terminal point of an arc if we know the length of the arc on the unit circle. Thinking in terms of functions, we want to be able to input an arc length *t*, then get out the coordinates (*x*, *y*) of the terminal point of the arc. That is also the same point where the terminal ray of the subtended angle intersects the circle. We will call this new function the wrapping function, because we are wrapping the tape measure around the circle for different arc lengths.

The wrapping function $W(t)$ maps the length of an arc that starts at (1, 0) on the unit circle to the coordinates of the endpoint of the arc.

Domain Range

 

1. In the following figure, an arc that measures $t=\frac{π}{2}$ is wrapped on the unit circle.
2. Label the coordinates of the initial point of the arc.

1. Label the coordinates of the terminal point of the arc.
2. Find $W\left(\frac{π}{2}\right)$.
3. Sketch in the arc that measures $-\frac{π}{2}$.
4. Find $W\left(-\frac{π}{2}\right)$.
5. Sketch the arc that measures $\frac{3π}{2}$.
6. Find $W\left(\frac{3π}{2}\right)$.

**Exploring the Periodicity of the Wrapping Function**

A function is called “periodic” if its values repeat at regular intervals. Graphically this means if you shift the graph of *f* horizontally by *p* units (where *p* is the period) that the new graph will look just like the old graph. The period is how long it takes until the pattern is repeated. If a function is periodic there will be many horizontal shift intervals that will result in an identical graph. The period is the smallest positive value of *p* that yields $f(t+p)= f(t).$

1. On the unit circle below:
2. Sketch the arc that measures $t=\frac{5π}{2}$.
3. Find $W(\frac{5π}{2})$.
4. Sketch the arc that measures $t=\frac{9π}{2}$.
5. Find $W(\frac{9π}{2})$.
6. Find another value for *t*, not yet mentioned, such that the output of the wrapping function is (0,1).



The output of the wrapping function repeats itself every 2$π$ so the period is 2$π$. See how adding $2π$ to the arc length does not change the coordinate of the endpoint of the arc.

$$W\left(\frac{π}{2}\right)=\left(0,1\right) $$

$W\left(\frac{π}{2}+2π\right)=W\left(\frac{5π}{2}\right)=\left(0,1\right) $

$W\left(\frac{π}{2}+2(2π)\right)=W\left(\frac{9π}{2}\right)=\left(0,1\right) $

$W\left(\frac{π}{2}+3(2π)\right) =W\left(\frac{13π}{2}\right)=\left(0,1\right) $

1. What is the next line in pattern?
2. Use the pattern to answer the following questions:
3. List the inputs to the wrapping function shown above.
4. How are the inputs changing?
5. How are the outputs changing?

Note that you get the same output for the function if you add or subtract any multiple of $2π$ to the input. $W\left(t\right)=W\left(t+k2π\right)$, where *k* is any integer. We say $2π$ is the period of the wrapping function. The period is the length of the interval on the horizontal axis (the independent variable) until the pattern repeats.

Definition:A function is *periodic* if $f(t)= f(t+period).$ Period > 0 and the smallest number that works.

**Exploring the Symmetry of the Wrapping Function**

Finding the output coordinates for $W\left(t\right)$ was straightforward for the arcs that are quarters of a circle. However, the other values are more difficult to find. $W\left(\frac{π}{4}\right)$ can be determined using the information that the *x*- and *y*-coordinates lie on the line $y=x$ and, therefore, are equal. The *x*- and *y*-coordinates are the lengths of the legs of an isosceles right triangle with hypotenuse 1 unit long. Different geometric equivalences can be used for the angles $\frac{π}{6}$ and $\frac{π}{3}$.

1. Complete the following:
2. Divide the following circle into eight equal arc lengths.
3. Fill in the blanks in the table. Two coordinates are done for you so you can use symmetry to find the coordinates of the other 7 equally spaced points on the circle.
4. On the unit circle, label each point for the wrapping function outputs that you determine below:



Use what you know about odd and even symmetry to complete the following:

$W\left(0∙\frac{π}{4}\right)=W\left(0 \right)= \\_\\_\\_\\_\\_\\_\\_\\_\\_$

$W\left(1∙\frac{π}{4}\right)=W\left(\frac{π}{4}\right)= \left(\frac{1}{\sqrt{2 }}, \frac{1}{\sqrt{2 }}\right)$$W\left(2∙\frac{π}{4}\right)=W\left(\frac{π}{2}\right)= \\_\\_\\_\\_\\_\\_\\_\\_\\_$

$W\left(3∙\frac{π}{4}\right)=W\left(\frac{3π}{4}\right)=\\_\\_\\_\\_\\_\\_\\_\\_\\_$

$W\left(4∙\frac{π}{4}\right)=W\left(π\right)= (-1,0)$

$W\left(5∙\frac{π}{4}\right)=W\left(\frac{5π}{4}\right)= \\_\\_\\_\\_\\_\\_\\_\\_\\_$

$W\left(6∙\frac{π}{4}\right)=W\left(\frac{3π}{2}\right)= \\_\\_\\_\\_\\_\\_\\_\\_\\_$

$W\left(7∙\frac{π}{4}\right)=W\left(\frac{7π}{4}\right)= \\_\\_\\_\\_\\_\\_\\_\\_\\_$

$W\left(8∙\frac{π}{4}\right)=W\left(2π\right)= \\_\\_\\_\\_\\_\\_\\_\\_\\_$

1. Given that $W\left(\frac{5π}{3}\right)=\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, find $W\left(\frac{4π}{3}\right)$.
2. Sketch the arcs $\frac{5π}{3}$ and $\frac{4π}{3} $on the unit circle. Label the coordinates of the endpoints of each arc.



**Part 2**

1. Take a moment to study the circle graphed below.
2. Since the radius is length 1, how far apart are the markings on the *x*-axis?
3. How far apart are the points on the circumference of the circle?
4. Just for practice, place a point at (.3,0) and place another point at the end of the arc that is 0.3 units long.



1. Locate the terminal point for an arc that measures 1 radian. It will be a point on the circle. Estimate the coordinates of this point. (Hint: to find the *x*-coordinate, drop a vertical line from the point down to the *x*-axis. Estimate the length of the line segment from the origin to the point on the *x*-axis. To find the *y*-coordinate, use the *y*-axis to estimate the length of the vertical line between the point at 1 radian and the *x*-axis.)

The coordinates are approximately\_\_\_\_\_\_\_\_\_\_\_\_\_\_, so W(1) ≈ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

1. Use the wrapping function to complete the table below:

|  |  |
| --- | --- |
| *t* | W(*t*) |
| 0.8 |  |
| 1.2 |  |
| 5 |  |
| -2.3 |  |
| 6.3 |  |
| 12.6 |  |
| 18.9 |  |