**Unit 7: Investigation 6 (3 Days)**

**Using Probability to Make Decisions**

**Common Core State Standards**

* S-MD-5. Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

a. Find the expected payoff for a game of chance. For example, find the expected winnings from a state lottery ticket or a game at a fast food restaurant.

b. Evaluate and compare strategies on the basis of expected values. For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.

* S-MD 6. Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).
* S-MD 7. Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game.)

**Overview**

This investigation begins with an introduction to expected value in the context of playing games of chance. Students discover that when comparing two games of chance, it is not just the probabilities that must be considered but also the payouts. After being presented with a formula for computing expected value, students evaluate several carnival games by calculating the average winnings per game (expected value) from the player’s point of view. Next, they apply what they have learned about expected value to the context of whether or not to purchase insurance or to guess on a multiple choice test. The situation of needing to randomly select answers to multiple choice questions leads to use of a calculator’s random number generator to choose the answer. The final activity focuses on the imperfect nature of drug testing and medical testing. Given the percentage of students who use performance enhancing substances and the reliability of the tests for detecting banned substance use, students discover that the probability that a randomly selected student is a substance user given test results are positive depends on the prevalence of substance use among students. The same techniques used in the banned substance use scenario are then applied to medical testing for Lyme disease.

**Assessment Activities**

**Evidence of Success: What Will Students Be Able to Do?**

* Determine the expected value of a probability model with numeric outcomes.
* In games of chance, use the expected value to decide which game is most advantageous for players.
* In situations of drug testing or medical testing, distinguish between true positive, false positive, true negative, and false negative test results.
* Calculate the probability that a person has a trait given that test results are positive for that trait. (Be able to informally apply Bayes Rule without having to use Bayes formula.)

**Assessment Strategies: How Will They Show What They Know?**

* **Exit Slip 7.6.1** asks students to create a probability model and then to use the model to calculate an expected value.
* **Exit Slip 7.6.2** gives students the percentage of people with a particular gene that is faulty and information on a genetic test. Students complete a tree diagram by filling in probabilities along the branches and then use this information to find the probability of a faulty gene given the test comes back positive.
* **Journal Entry** asks students to explain why it is important for insurance companies, after setting rates based on expected insurance payouts, to sell policies to a large number of people rather than to just a few people.

**Launch Notes**

Begin this investigation by asking the questions below. Some sample answers follow each question. However, students are not necessarily expected to get correct answers. What is important is the discussion of “expected value” on an informal level.

1. You roll a die 30 times. Sum the number of spots that land on the side facing up for each of the 30 rolls.

* What is the lowest sum you could get?

The lowest possible sum is 30, which occurs only if the side with one dot landed facing up on all 30 rolls.

* What is the highest sum you could get?

The highest possible sum is 180, which occurs only if the side with six dots landed facing up on all 30 rolls.

* What sum would you expect to get? Explain how you got your answer.

Sample answer: Each outcome 1, 2, 3, 4, 5, and 6 are equally likely. So, you would expect to get around 5 of each possible outcome for a total of 105, which is midway between 30 and 180.

2. Bet A pays $20 if you win and you have a probability of $\frac{1}{2} $of winning. Bet B pays $5,000 if you win and you have a probability of $\frac{1}{10}$ of winning.

* How much would you expect to win if you placed bet A 100 times? What would be the average winnings per bet?

You would expect to win 50 bets for a total winnings of $1,000. That would mean that you would win an average of $10 per bet.

* How much would you expect to win if you placed bet B 100 times? What would be the average winnings per bet?

You would expect to win 10 bets for a total winnings of $50,000. That would mean that you would win an average of $500 per bet.

* In placing 100 bets, would you be better off placing bet A or bet B?

You would be better off placing bet B.

**Teaching Strategies**

In the launch activity, students were asked about “expected value” before being formally introduced to the concept. In situation (2), students would very likely choose bet *B* even though *A* offers a better chance to win. It would be foolish to decide which bet to make just on the basis of the probability of winning. How much you can win is also important. When a random process has numerical outcomes, we are concerned with their amounts as well as with their probabilities.

Returning to situation 2, it should be noted that players generally are required to put money down when they place their bets otherwise there is no risk in betting. If a player does not win, then the player forfeits the money he/she put down.

**Activity 7.6.1 Should You Play?**continues with situation (2) from the launch discussion but requires that money be put down when making each bet. Students determine the expected winnings for this modification before learning a formula for calculating expected value from a probability model. After students are given a formula for computing expected value, they turn their attention to finding the expected values of a variety of carnival games. Students use the expected values to determine which games are better from the player’s point of view. In addition, they determine the expected earnings of the carnival.

**Group Activity**

**Activity 7.6.1** is designed as a group activity. Students should work on *Part I: Analyzing Two Bets* as a whole class. After completing Part I, challenge students to come up with their own formula for computing expected value from a probability model with a finite number of outcomes. Then present the formula given at the end of Part I. Students should work on *Part II: Senior Class Sponsored Carnival* in small groups of 2 or 3 students.

 **Activity 7.6.1** consists of two parts. Part I should be a whole-class activity. Give students a few minutes to work on Part I individually, before discussing the solution as a class. At the end of Part I, a formula is presented for computing an expected value given a finite probability model in which outcomes are numeric. This formula is given below.

**Expected Value or Mean Value**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Outcome |  |  | ... |  |
| Probability |  |  | ... |  |

The expected value is the sum of the outcomes times their probabilities. This is expressed by the following formula:

 Expected value = $x\_{1}p\_{1}+ x\_{2}p\_{2}+ \cdots +x\_{n}p\_{n}$

You may want to present an example of computing an expected value from the formula before students begin Part II. If that is the case, here’s an example you could use.

**Example:** Suppose you pay $10 to play a game in which you roll a die. If you roll a 4, 5, or 6, then you are paid, $10, $20, or $40, respectively. The table below is a probability model for your net winnings from playing this game. (You’ll have to subtract the price of playing from any money you are paid.)

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Outcome of roll | 1 | 2 | 3 | 4 | 5 | 6 |
| Net winnings | –$10 | –$10 | –$10 | 0 | $10 | $30 |
| Probability | $$\frac{1}{6}$$ | $$\frac{1}{6}$$ | $$\frac{1}{6}$$ | $$\frac{1}{6}$$ | $$\frac{1}{6}$$ | $$\frac{1}{6}$$ |

Expected net winnings =
(–$10)($ \frac{1}{6}$) + (–$10)($ \frac{1}{6}$) + (–$10)($ \frac{1}{6}$) + ($0)($ \frac{1}{6}$) + ($10)($ \frac{1}{6}$) + ($30)($ \frac{1}{6}$) ≈ $1.67

Conclusion: If you played this game many, many times, on average you would win $1.67 per game.

**Differentiated Instruction (For Learners Needing More Help)**

If students struggle with the formula for expected value, have them add another row to the probability model table. In that row, they should enter the product of the outcome and its corresponding probability, . Then, to find the expected value, they need only sum the values in the added row of their table.

**Differentiated Instruction (Enrichment)**

Challenge students to create one or two additional carnival games for **Activity 7.6.1**. Students should determine the expected value of each of their games. In addition, they should determine how much more the senior class could expect to earn if each of their games were played 100 times by carnival attendees.

**Exit Slip 7.6.1** can be assigned after students have completed **Activity 7.6.1** or **Activity** **7.6.2**.

**Activity 7.6.2 Using Probability for Decisions and Random Selection** presents real-world situations (not involving games of chance) in which finding the expected value is useful for decision making. The contexts include buying insurance for a cell phone or washing machine and guessing answers on SAT exams. The last few questions focus on using a calculator’s random number generator to make random selections.

 If you don’t have class time for **Activity 7.6.2,** it could be assigned as homework.

**Journal Entry**

The idea of insurance is that we all face risks that are unlikely but carry a high cost. Think of a fire destroying your home. Insurance spreads the risk – we all pay a small amount and the insurance policy pays a large amount to those few of us whose homes burn down. Suppose that an insurance company looks at records for millions of homeowners and sees that the expected loss from fire in a year is $250 per homeowner. The company plans to sell fire insurance for $250 plus enough to cover its costs and profit. Explain clearly why it would be unwise to sell only 12 policies. Then explain why selling thousands of such policies is a safe business.

Look for reasoning similar to the following. Students should note that the expected value is the long run average cost to the insurance company. If the expected cost to an insurance company is $250 for fire damage, then that average cost is spread out over many, many policies sold by that insurance company. If a new company sells only 12 fire insurance policies, then the new company is putting itself at great risk – if the homeowner of one of its policies loses his/her home in a fire, then the insurance company will have to pay the homeowner the cost of the home, which may be $100,000 or more. The $250 plus the profit on the other 11 policies will not cover that amount of payout. However, if many thousands of policies are sold and one homeowner loses a house due to fire, then the $250 plus the profit from the many other policies will cover the cost of the insurance company’s payout.

**Activity 7.6.3 Imperfect Testing** begins witha hands-on simulation of testing for performance enhancing substances. Hershey’s Kisses, almond and plain, represent users and non-users of performance enhancing drugs, respectively. All Kisses are covered in aluminum foil so that you can’t tell the users from the non-users. Each Kiss is tied with a silver or gold ribbon that designates the outcome of the drug test. Silver ribbons indicate a negative test result and gold ribbons indicate a positive test result. The simulation is designed to help students differentiate between true positive and false positive test results. Next, students are given information about the percentage of students who are banned substance users, as well as information on the accuracy of the test. Students are asked to determine the likelihood that a randomly selected student who tests positive for banned substance use is actually a substance user. At the end of the activity, the context switches to testing for Lyme disease.

**Group Activity**

**Activity 7.6.3** begins with the Kiss simulation of testing for performance enhancing drugs. This is designed to be a whole class activity. Students should work through questions 1–3 in small groups of 2 or 3 students. If short on time, questions 4 and 5 can be assigned for homework.

**Materials for** **Activity 7.6.3**:

* A bag of Hershey’s plain milk chocolate Kisses and Hershey’s almond Kisses. (The milk chocolate Kisses are covered in silver wrap and the almond Kisses are covered in gold wrap.)
* Small squares of aluminum foil sufficiently large to cover a Kiss leaving a foil tail.
* Silver (or white) ribbon and gold (or yellow) ribbon.
* Opaque container for Kisses

**Preparation for Activity 7.6.3:**

* You will ive one Hershey’s Kiss per student. Roughly 30% of the Kisses should contain almonds and the other 70% should be plain milk chocolate. (For example, if there are 24 students in class, you should have 7 almond Kisses and 17 plain Kisses.)
* **Preparation of almond Kisses:** Use small squares of aluminum foil to cover the almond Kisses. Pull the wrap together at the top of the Kiss forming a tail. On around 90% of the Kisses, tie a gold (or yellow) ribbon near the base of the aluminum foil tail. On the remaining foil-covered almond Kisses, tie a silver (or white) ribbon.
* **Preparation of plain Kisses:** Use small squares of aluminum foil to cover the regular milk chocolate Kisses. On around 80% of these Kisses, tie a silver (or white) ribbon. On the remaining foil-covered regular Kisses, tie a gold (or yellow) ribbon.
* When you have finished covering the Kisses with aluminum foil and tying a ribbon around the base of the foil tail, put the Kisses into an opaque container.

**Kiss Simulation**

 Each student should reach into the container without looking, and draw out a Kiss. Then students should record in a table similar to the one below the number of positive test results (Kisses tied with gold (or yellow) ribbon) and negative test results (Kisses tied with silver (or white) ribbon.

|  |  |  |  |
| --- | --- | --- | --- |
|  | Total number | Substance users | Not substance users |
| Positive tests |  |  |  |
| Negative tests |  |  |  |

Students with positive test results should unwrap the outer foil on their Kisses. Then, record the number of students with almond Kisses (gold wrapping, substance users) and regular Kisses (silver wrapping, not substance users) in the last two entries of the first row of the table. Next, students with negative test results should unwrap the outer foil on their Kisses. Then, record the number of students with almond Kisses (gold wrapping, substance users) and regular Kisses (silver wrapping, not substance users) in last two entries of the second row of the table. Ask students to classify their situation as a true positive, false positive, true negative or false negative. Discuss question 1 as a whole class.

After the Kiss simulation, students should work on questions 2 and 3 in small groups. Discuss the answers to 2(e) and 3(e). Students should note that when the percentage of banned substance users is low in the student population, then the percentage of actual users among the positive test results is also low. In other words, a high percentage of the positive test results are false positives.

Questions 4 and 5 can be assigned for homework. Students apply the same techniques used in questions 2 and 3 but for a different context, testing for Lyme disease.

The mathematics involved in **Activity 7.6.3** can be summarized by a rule known as **Bayes Rule** (or Bayes Theorem). Bayes Rule provides a connection between *P*(*A*|*B*) and *P*(*B*|*A*).

**Bayes Rule**



**Exit Slip 7.6.2** should be assigned after students complete Activity 7.6.3.

Students should continue to work on the Performance Task that was started in Investigation 5.

**Closure Notes**

This Investigation focused on how probability might be used to make decisions. First, students learned how to calculate the expected value of a random process that had a finite number of possible numeric outcomes. Based on the expected value of a game of chance, students were able to determine whether that game was a good game to play. This same idea carried over to the analysis of whether or not to purchase insurance. Then we switched gears and looked into tests for traits such as tests for use of performance enhancing substances or specific diseases. Unfortunately drug tests, medical tests, or more generally, tests for particular traits are not perfect. In fact, a positive test result does not always mean that the trait is likely. This was part of the reason why Texas stopped randomly testing high school student athletes for performance enhancing substances.

**Vocabulary**

**Bayes Rule:** 

**Expected value**: The long-run average value of a random process with numeric outcomes over many, many trials.

**False negative:** A test result reports that a trait is absent when the trait is present.

**False positive:** A test result reports a trait is present when the trait is absent.

**True negative:** A test result reports that a trait is absent when the trait is absent.

**True positive:** A test result reports the presence of some trait when the trait is present.

**Resources and Materials**

Activity 7.6.1 Should You Play?

Activity 7.6.2 Using Probability for Decisions and Random Selection

Activity 7.6.3 Imperfect Testing

Exit Slip 7.6.1

Exit Slip 7.6.2

Journal Entry

Materials

Bag Hershey’s plain Kisses

Bag Hershey’s almond Kisses

Aluminum foil

Silver (or white) ribbon

Gold (or yellow) ribbon

Opaque container to hold Kisses