**Unit 5: Investigation 2 ( 2 -3 Days)**

**Natural logarithms and base *e***

**Common Core State Standards**

F.BF.5 Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

A.REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous roots may arise.

**Overview**

In this Investigation, the concepts of investigation 1 are extended by using the notion of finite compounded interest to develop *e*, the base of the natural logarithm. Students studied finite compound interest in Algebra 1 Investigation 7 in Activity 7.5.5. If your students have not done that activity they should be assigned that activity or an equivalent one so they first compute some compound interest for a finite number of compounding periods before they do some work with continuous compounding. Students will also explore *e* graphically by graphing y = 2x , y = ex and y = 3x and zooming in near the y intercept and computing the slope of each curve near (0, 1). Students summarize that as n increases without bound and , but with informal notation. Students will write equations of the form to model continuous exponential growth, will solve for *t* by taking the logarithm of both sides, using the power rule for logs and obtaining and will use technology to approximate the answer where appropriate. They will model some applied problems.

**Assessment Activities**

**Evidence of Success: What Will Students Be Able to Do?**

* Model compound interest with an appropriate formula.
* Distinguish between situations that need finite compound interest and those that need continuous compounding.
* Approximate ln *c* with an integer and using technology obtain a decimal approximation.
* Solve by writing as and then approximating *x* with technology; recognize that for the equation has no solution.
* Solve for *t* by rewriting as and approximating appropriately with technology.
* Solve some applied problems such as finding a doubling time, half-life and other problems including financial ones.

**Assessment Strategies: How Will They Show What They Know?**

* **Exit Slip 5.2.1** has students use both the finite compound interest formula and the continuous compounding formula.
* **Exit slip 5.2.2** has students solve equations that use the logarithmbase 10 and the natural logarithm and asks them to solve one application problem.
* **Journal Prompt 1** How does *e* relate to compound interest?
* **Activity 5.2.1** **How Many Compounding Periods Should I Try to Get ?** This activity begins with finite compound interest and numerically makes a case for the existence of the number *e* and introduces continuous compounding.
* **Activity 5.2.2 Revisiting *e* and Compound interest** may be omitted if students were able to master the new material in Activity 5.2.1 and do not need further practice with compound interest. It can also in whole or part be used as a homework assignment after Activity 5.2.1 has been discussed.
* **Activity 5.2.3The Remarkable *e*** looks at *e* using a graphical approach and a series approach.
* **Activity 5.2.4 Equations Involving Logarithms** has students solving equations that need logarithms or have logarithms in the equation itself and applied problems that rely upon continuous compounding and other continuous applications.

**Launch Notes**

Have students consider that their parents have won 10 million dollars in a state lottery and they want to invest wisely. If students have not had unit 7 in Algebra 1, time should be spent developing the finite compound interest formula. See Activity 7.5.5 in Algebra 1. Assume the annual interest rate is 12% for ease of computation. To compound twice a year, their parents will get 6% every 6 months. To see a pattern, compute for a second year.

$10,000,000 + .06($10,000,000)= (1 + .06)(10,000,000) for the first 6 months

$(1 + .06)(10,000,000)+ .06(1 + .06)(10,000,000)=(1 +.06) (1 + .06)(10,000,000)= (1 + .06)2(10,000,000) for year 1

(1 + .06)2(10,000,000) + .06(1 + .06)2(10,000,000)=(1+.06) (1 + .06)2(10,000,000)= (1 + .06)3(10,000,000) for 1.5 years

(1 + .06)3(10,000,000) + .06(1 + .06)3(10,000,000)= (1 + .06)4(10,000,000) for 2 years

Extend until students note that the rate will be *r/n* where *r* is the rate expressed as a decimal, *t* is the number of years, *n* is the number of compounding periods, the total number of compounding periods will be *nt*, and the amount at the end of *t* years (1 + .06)*nt*(10,000,000).

**Teaching Strategies**

Following the development of the finite compound interest formula the teacher may want to do a few problems from **Activity 5.2.1** in whole class mode, and then divide the class in groups to work through the exercise until they have completed all but the last row of the table in number 8.

Be sure to review simple interest and how it differs from finite compound interest. Then use the results of the table to develop the continuously compounding interest formula A= P*e*rt.

**Group Activity 5.2.1** Two students, working with their calculators, should create a table for values of as *n* increases from 1 (compounded once each year) to monthly, weekly, daily, hourly and even to compounding by minutes and seconds. It is critical here that students write down all decimal places possible to see the pattern in the outcomes.

When all students have completed their tables, a class discussion of the results should be encouraged. Students should note that as the number of times of compounding increased in a year, the rate of growth slowed down and seems to approach a number (limit). This special number is called *e*. Jacob Bernoulli was the first to point out the connection between continuous compound interest and (1+ 1/*n*)*n* as *n* gets larger. He used the Binomial Theorem to show that the limit as *n* got larger and larger had to be between 2 and 3. Euler named it ‘*e’*. The use of the letter *e* appeared in a letter he wrote in 1731. Euler showed that

 *e* = 1 + 1/1+1/[1(2)+1/[1(2)(3)]+1/[1(2)(3)(4)] …

and **Activity 5.2.3** will use this series to approximate *e*. Like *e* is an irrational number which means it cannot be written as a ratio of integers. Also like  *e* is transcendental which means it is not the root of any polynomial equation with integer coefficients. Emphasize that *e* is not equal to 2.71828 but rather *e* is approximated by 2.71828.The decimal expansion of *e* is known to 1 trillion places of accuracy! Note: *e* looks like it is repeating but it is not.

**Differentiated Instruction (For Learners Needing More Help)**  Have students explicitly show their computations for the change of units such as n = the number of hours in a day times the number of days in a year as they work through Activity 5.2.1.

**Differentiated Instruction (For Learners Needing More Help)**

Emphasize that r is the decimal representation of the percentage given; practice changing a percent to a decimal and a decimal to a percent.

**Activity 5.2.2** can be used for homework for students needing more practice after completing Activity 5.2.1. In the activity students will again create a table for to

discover the special irrational (and transcendental) number *e* and will use the results of a second table to verify the continuously compounding interest formula A= P0 *e*rt.

**Exit Slip 5.2.1** can be used any time after Activity 5.2.1 or Activity 5.2.2, if it was assigned.

**Differentiated Instruction (Enrichment)** Have students research the number *e*. Unlike π, mankind was only aware of its existence about 400 years ago. What kind of number is it? Rational, irrational? Is it a special irrational? How many irrationals are there like π and *e*? How are they different from √2 or √5? Students will hopefully in their research uncover the fact that it is real and it is irrational. But unlike√2 which is a solution of a polynomial equation with integer coefficients, *e* is irrational and nonalgebraic. It is transcendental meaning it goes beyond algebraic numbers . π is also nonalgebraic.

 **Journal Prompt 1** How does *e* relate to compound interest? Students should include in their explanation the need to have many, many compounding periods and when one does, that (1+ 1/*n*)*n*as *n* gets larger approaches *e* and the finite compounding formula becomes P = P0 *e*rt.

**In Activity 5.2.3**, students will see *e* in a few interesting places. The YouTube video: The Number *e* Is Everywhere by Randell Heyman [www.youtube.com/watch?v=b-MZumdfbt8](http://www.youtube.com/watch?v=b-MZumdfbt8) is well worth the 3 minutes of class time it uses for viewing. The web site http://www-history.mcs.st-and.ac.uk/PrintHT/e.html has a four page printout filled with readable interesting tidbits. It is not necessary, if time does not permit, to do all the examples in this activity. Numbers 2 and 3 will reappear if students take more math and should be done if time permits. Select one or two tidbits of interest to your class.

In our previous Investigation 1 of this Unit, we found the logarithm function defined by is the inverse of the exponential function. Consider the special case of an exponential function with base, . This exponential function is important in mathematics and science and is called the natural exponential function. Its inverse, , is called the natural logarithmic function and is more commonly written as . A suggested group activity appears below.

**Group Activity** On calculators or equivalent technology, graph . Write down properties of this function including domain, range, intercepts and end behavior. Rewrite the properties we found in Activities 5.1.3 and 5.1.4 in terms of natural logarithm.

In Activity 5.1.5, we verified the two properties and . Students should rewrite these two properties with . In **Activity 5.2.4** students will solve a variety of logarithmic and exponential equations and some application problems. They will use equation , and solve for *x*, by dividing both sides by *a* and *writing* in “log form,” , and then finally dividing both sides by *c*. Students will use the properties that follow from the fact that the logarithmic and exponential function families are one-to-one . The reasonableness of the properties stated below needs to be discussed.

**If A, B and b are positive real numbers and b ≠ 1, the logbA = logb B if and only if A = B.**

**If a > 0 and a ≠ 1, then ax = ay if and only if x = y.**

Now students have another way to generate equivalent equations. They can take the logarithm of both sides of the equation or use the fact that if two powers are equal but have the same base that the exponents must be equal. Applications for exponential growth and decay are provided. The **Exit Slip 5.2.2** can be distributed any time once **Activity 5.4.4** has been discussed.

**Differentiated Instruction (Enrichment)**

In pairs, student should graph y = 1/x using a scale of 10 blocks equals one unit. Using technology they should make a table for inputs of 1 to 2.8 by 0.1 and the associated outputs, carefully plot the points and draw a smooth curve through them. They should graph the vertical lines x = 1 and x = *e*. They should see which pair of students comes closest to having the equivalent of 100 little blocks under the curve. In the calculus they will be able to show that the area under the curve and between x = 1 and x = *e* is exactly 1 square unit.

**Closure Notes**

Students could each write down on a 3 by 5 card one fact or theorem or definition they learned in this investigation and then these can be collected and shared**.** The teacher might want to have a few cards filled out with the essential concepts in case these are not mentioned but it is the students remembering tidbits that will make this activity fun.

**Vocabulary**

Amount

Annual Interest rate

Certificate of Deposit (CD)

Continuous growth

Continuously compounded interest

Finite compound interest

Future Value

Interest

Principal

Simple interest

**Resources and Materials**

**Activity 5.2.1 and 5.2.4 must be completed**. **Activity 5.2.2 can be skipped or just in part be assigned if the class clearly understood after completing Activity 5.2.1 that as *n* gets large is close to *e* and as *n* gets larger approaches *e*r, and how to use the formula A = P *e*rt. If time is tight select just one or two of the interesting tidbits about *e* in Activity 5.2.3.**

Activity 5.2.1 How Many Compounding Periods Should I Try to Get?

Activity 5.2.2 Revisiting e and Compound interest

Activity 5.2.3The Remarkable e

Activity 5.2.4 Equations Involving Logarithms

TI 84 graphing calculator or similar technology

Graph paper

Internet access

3 by 5 cards

[www.youtube.com/watch?v=b-MZumdfbt8](http://www.youtube.com/wartch?v=b-MZumdfbt8)

<http://www-history.mcs.st-and.ac.uk/PrintHT/e.html>