**Unit 3: Investigation 1 (3 Days)**

**Sums of Interior Angles of Polygons**

**Common Core State Standards**

* G-CO.10. Prove theorems about triangles. Theorems include: *measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length;**the medians of a triangle meet at a point.*

**Overview**

Students will use transformations to prove that the sum of the interior angles of a triangle is 180°. They will then show that the sum of the interior angles of a convex quadrilateral is 360°, by using a diagonal to form two triangles. They will generalize to all convex polygons with *n* sides, where the sum of interior angles is (*n*–2)180°. The AAS Congruence Theorem will be introduced as a consequence of the Triangle Sum Theorem.

**Assessment Activities**

**Evidence of Success: What Will Students Be Able to Do?**

* Understand and explain the reasoning behind proofs of the Triangle Sum, Quadrilateral Sum and Polygon Sum Theorems
* Apply the above theorems to find unknown angle measures
* Prove and apply the AAS Congruence Theorem

**Assessment Strategies: How Will They Show What They Know?**

* **Exit Slip 3.1.1** asks students to identify pairs of congruent angles in the proof of the Triangle Sum Theorem
* **Exit Slip 3.1.2** asks students to apply the AAS Congruence Theorem
* **Journal Entry** asks students to describe how the Polygon Sum Theorem is related to the Triangle Sum Theorem.

**Launch Notes**

Most students should be familiar with the triangle sum property; in fact, the Common Core 8th grade standards (8-G 5) require students to see an “informal” proof of the theorem. Nevertheless assume from the start that this property was never learned or has been forgotten. Use **Activity 3.1.1** **Sums of Interior Angles** to help students discover the property using a variety of tools. Have each student draw a different triangle, measure the interior angles with a protractor, and compute the sum. Expect sums to range from 178° to 182°. Explain to the class that the deviation from 180° is likely due to the limited accuracy of their measuring instrument. Then demonstrate using Geogebra or Geometer’s Sketchpad that the sum of the angles of a variety of triangles appears to be 180°. Point out that the computer can measure much more accurately but still only to the nearest hundredth or thousandth of a degree. So the best we can say from measurement is that the sum of the angles is close to 180°; we cannot say with certainty that it is exactly 180°. However, at this point, based on the evidence we have examined we can formula the **Triangle Sum Conjecture**: The sum of the interior angles of every triangle is 180°.

**Group Activity**

Have students work on **Activity 3.1.1** in heterogeneous groups. They can check each other’s protractor measurements and resolve any discrepancies. Students who are having trouble reading a protractor will receive help from their more able peers.

You may then want to demonstrate the triangle sum conjecture by tearing off three corners of a triangle and arranging them to form a straight angle. This can be done with the triangles the students have drawn if you give them scissors to cut them out. Or you may want to do it yourself as a classroom demonstration.

(A video showing this is found at <http://www.brightstorm.com/math/geometry/triangles/triangle-angle-sum/>. You may want to show just the first part (1 minute, 45 seconds) since we’ll be looking at a slightly different proof a little later in this investigation. You may also want to look at an NCTM illuminations activity that demonstrates both the triangle and quadrilateral sum properties: <http://illuminations.nctm.org/Activity.aspx?id=3546>.)

Ask students whether they find the demonstration with the cut-out angles convincing. Most of them will probably say “yes.” Sow some doubt in their mind by asking “How can we be absolutely sure that the sides of these two angles actually form a straight line? It may look like it is a straight line, but is it really?” This will motivate the need for a proof of the triangle sum conjecture.

Repeat the process for quadrilaterals. Encourage students to draw quadrilaterals with no parallel sides. When they measure the angles and find the sum, their results should be within 2° or 3° of 360°. Based on this experiment the class may formulate the Quadrilateral Sum Conjecture: The sum of the interior angles of a quadrilateral is 360°. The issue of whether this applies to non-convex quadrilaterals may be deferred until later in the investigation, unless a student happens to draw a non-convex quadrilateral. When they cut out all four corners and put the angles together they should appear to fill the space around one vertex, thus suggesting that the sum is 360°. As with the triangle demonstration you should point out that this demonstration does not constitute a proof.

**Teaching Strategies**

Having introduced students to the Triangle Sum and Quadrilateral Sum Conjectures you have now motivated students to prove these conjectures as theorems.

**Activity 3.1.2 Proving the Triangle Sum Theorem** is based on a transformational approach. You may want to discuss the first part of the proof with the entire class. Once it has been established that *C’BA’* is a straight line, students should be able to complete the rest of the proof. You may use any format—two column, flow proof, or paragraph proof—but be sure students are able to justify every step. There are two versions of this activity. **Activity 3.1.2a** has students fill in the blanks; **Activity 3.1.2b** is more open ended. See a completed proof of this theorem in the Appendix.

The proof of the Triangle Sum Theorem relies on our ability to add the measures of adjacent angles. You may want to introduce the **Angle Addition Postulate**, which states that when two angles share a vertex and a common side that lies between them, the sum of their angle measures is equal to the measure of the angle formed by their non-common sides. This is easier to visualize than to express in words. In the figure on the left, m$∠BAC= m∠BAD+ m∠DAC$.

This definition extends to cases where the sum of the two angles is greater than 180° as shown in the figure on the right, provided we consider $∠BAC$ to be a reflex angle. When the sum is equal to 180° the angles form a linear pair. Thus the Angle Addition Postulate is a generalization of the Linear Pair Postulate.

****

In the proof of the Triangle Sum Theorem we have three angles sharing a vertex, so we are actually applying the **Angle Addition Postulate** twice.

**Exit Slip 3.1.1** may be given any time after **Activity 3.1.2** has been completed.

**Activity 3.1.3 Proving the Quadrilateral Sum Theorem** is based on drawing a diagonal to partition the interior of a quadrilateral into two triangles. Students should be able to justify each step. Again, there are two versions of this activity. **Activity 3.1.3a** has students fill in the blanks; **Activity 3.1.3b** is more open ended.

See **Note to Teachers on Non-convex Polygons** in the Appendix.

**Differentiated Instruction (Enrichment)**

Have students explore the question of whether the Quadrilateral Sum Theorem applies to non-convex polygons.

Generalizing toa polygon with *n* sides to find the sum of interior angles involves an informal application of the principle of mathematical induction. In **Activity 3.1.4 The Polygon Sum Theorem** students will see that a diagonal may be drawn to partition the interior of a pentagon into a quadrilateral and a triangle. Similarly the interior of a hexagon may be partitioned into a pentagon and a triangle. Students then may fill in the table that relates the number of sides to the sum of the interior angles. In each case they add 180° to the previous sum. This is a recursive rule similar to those introduced in Unit 1 of the Connecticut Core Algebra 1 Curriculum. By graphing the data they should recognize a linear relationship. From that they should be able to derive the function *f*(*n*) = 180*n* – 360 or *f*(*n*) = 180(*n* – 2).

**Differentiated Instruction (For Learners Needing More Help)**An alternative proof may be more appealing to some students. Draw all diagonals of the polygon from one vertex and show that *n* – 2 triangles are formed. Since the sum of the interior angles of each triangle is 180° the total for the angles of the polygon is (*n*– 2)180°. You may need to remind students of the commutative property of multiplication: (*n*– 2)180 = 180(*n* – 2).

**Differentiated Instruction (Enrichment)**

Students may extend the reasoning used in questions 3–6 of Activity 3.1.4 to prove the general case for a convex polygon with *n* sides using mathematical induction. The key idea is to show that a convex polygon with *k* + 1 sides may be subdivided by a diagonal into a polygon with *k* sides and a triangle. If we already know that the sum of the interior angle of the polygon with *k* sides is (*k*– 2)180°, we can then show that the angle sum for the polygon with *k* + 1 sides is
(*k* + 1– 2)180°.

**Activity 3.1.5 Applications of the Triangle, Quadrilateral and Polygon Sum Theorems** gives students practice using the theorems they have just proved to find the unknown measures of angles given sufficient information. This may be an appropriate homework assignment. If you only discuss triangles on the first day, the first part of the activity may be assigned the first day and the rest the following day.

In **Activity 3.1.6 The AAS Congruence Theorem** students learn that if two pairs of angles from two triangles are congruent, the third pair of angles must also be congruent. This is a consequence of the Triangle Sum Theorem. From this result the AAS Congruence Theorem follows. Students are then given several exercises in which all four congruence theorems learned so far (SAS, SAS, SSS, and AAS) may be applied.

**Note to teachers:** Our proof of the AAS Congruence Theorem relies on the Triangle Sum Theorem, which in turn relies on the Euclidean Parallel Postulate. Euclid and others prove AAS without relying on the Parallel Postulate. It is a theorem of neutral geometry as it holds in hyperbolic as well as Euclidean geometry.

**Exit Slip 3.1.2** may be given following **Activity 3.1.6.**

**Journal Entry**
How is the sum of the angles of any polygon related to the sum of the angles of a triangle? Look for students to explain how diagonals of a polygon subdivide the polygon into figures with fewer sides.

**Closure Notes**

Summarize the Investigation by making a chart showing the dependency of the theorems on one another. (Show links from the Triangle Sum Theorem to the Quadrilateral Sum Theorem, Polygon Sum Theorem, and the Third Angles Theorem. Show a link from the Quadrilateral Sum Theorem to the Polygon Sum Theorem and a link from the Third Angles Theorem to the AAS Congruence Theorem.)

**Vocabulary**

Interior angle of a polygon

(Note: when we speak of the “angles” of a polygon, we refer to the interior angles. Exterior angles will be discussed in the next investigation.)

Diagonal

Convex polygon

Non-convex polygon

**Theorems**

**Triangle Sum Theorem:** In any triangle the sum of the interior angles is 180°.

**Quadrilateral Sum Theorem:** In any convex quadrilateral the sum of the interior angles is 360°.

**Polygon Sum Theorem:** In any convex polygon with *n* sides, the sum of the interior angles is 180°(*n* – 2).

**Third Angles Theorem:** If two angles of one triangle are congruent to two angles of another triangle, then the third pair of angles are also congruent.

**AAS Congruence Theorem:** It two angles and a non-included side of one triangle are congruent to two angles and the corresponding non-included side of a second triangle, then the two triangles are congruent.

**Resources and Materials**

Activity 3.1.1 Sums of Interior Angles

Activity 3.1.2 Proving the Triangle Sum Theorem

Activity 3.1.3 Proving the Quadrilateral Sum Theorem

Activity 3.1.4 The Polygon Sum Theorem

Activity 3.1.5 Applications of the Triangle, Quadrilateral and Polygon Sum Theorems

Activity 3.1.6 The AAS Congruence Theorem

Tools: straight edge, protractor, scissors (optional)

Software: Geometer’s Sketchpad or Geogebra

Web sites:

<http://www.brightstorm.com/math/geometry/triangles/triangle-angle-sum/>.

<http://illuminations.nctm.org/Activity.aspx?id=3546>.

**Appendix**

**Note to Teachers on Non-convex polygons**: When attempting to verify the quadrilateral sum theorem with a non-convex quadrilateral you may run into some difficulty, depending upon how angles are measured.

Observe what happens when we measure angles in Geometer’s Sketchpad.



Now observe what happens with Geogebra.





The difference is that Geogebra allows for the measurement of angles greater than 180° whereas Geometer’s Sketchpad does not. And yet the proof for the Quadrilateral Sum Theorem is valid if the diagonal is drawn from B to D in the non-convex quadrilaterals on the right, provided we allow the sum of the two angles formed at B to exceed 180°.

To avoid this difficulty we usually restrict the Quadrilateral Sum and Polygon Sum Theorems to convex polygons. Your more able students may want to explore extending the theorems to include non-convex polygons.