**Activity 3.3.5 Identities Galore: Summing It Up!**

Michael Hartley recounts a famous story online that you can find at <http://www.dr-mikes-math-games-for-kids.com/rice-and-chessboard.html>. The story goes like this:

There's a famous legend about the origin of chess that goes like this. When the inventor of the game showed it to the emperor of India, the emperor was so impressed by the new game, that he said to the man, "*Name your reward!*"

The man responded, "*Oh emperor, my wishes are simple. I only wish for this. Give me one grain of rice for the first square of the chessboard, two grains for the next square, four for the next, eight for the next and so on for all 64 squares, with each square having double the number of grains as the square before.*"

The emperor agreed, amazed that the man had asked for such a small reward - or so he thought. After a week, his treasurer came back and informed him that the reward would add up to an astronomical sum, far greater than all the rice that could conceivably be produced in many, many centuries!

What’s the mathematics behind what made the inventor’s request total up to all that rice?

You have seen a variation of this story in Algebra 1 and you will return to a variation of it in Unit 5 of this course. Each time we see the story we are able to examine it from a different perspective. It turns out that we can investigate the mathematics behind the solution to the rice problem using polynomials. Yes polynomials not the exponential family!

**I A New Identity:**

Examine the following products and see if you can determine a pattern that you can generalize.

1. *(x–1)(x+1)*
2. *(x–1)(x2 + x + 1)*
3. *(x–1)(x3 + x2 + x + 1)*
4. *(x–1)(x4+ x3 + x2 + x + 1)*
5. Based on the pattern you see in problems 1-4, what will the product be for:

*(x–1)(x9+ x8 + x7 + • • • + x2 + x + 1)*

Create a general algebraic identity involving this pattern.

The identity found by following the pattern above is that:

 *(x–1)(xn+ xn–1 + xn–2 + • • • + x2 + x + 1) = xn+1 – 1 • • • • • (1)*

Dividing both sides by (x–1), we obtain an equivalent form of the identity:

 $\left(x^{n}+ x^{n-1} + x^{n-2} + • • • + x^{2} + x + 1\right)=\frac{x^{n+1}–1}{x-1}$ • • • • • (2)

**II Use the identity to solve the rice problem.**

1. Write the expression for the amount of rice the inventor of the game of chess received.

2. Apply equation (2) to the solution you found in part 1. How many grains of rice did the inventor receive? Use your calculator to represent the sum.

3. Use the same technique to calculate the sum for the following expression:

 $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+••••••+\frac{1}{2^{10}}$

|  |  |
| --- | --- |
| 4. What would this sum approximate if you continued the sum out to 100 terms? n places? What is this sum approaching as n approaches ∞? (Hint: Use the figure to the right to help you justify your response.) |  |

5. Can this identity be extended to factor an – bn? Make a conjecture about what this binomial can factor into and check your conjecture for n = 3?

6. Use your conjecture in #5 to factor 32x5 – y10. Multiply out the results to check your work.

**A Similar Identity:**

Another identity similar to the one you just found can also be established. Calculate the following products to see if you can create another identity to factor polynomials.

1. *(x+1)(x2 – x + 1)*
2. *(x+1)(x4 – x3 + x2 – x + 1)*
3. *(x+1)( x6 – x5 + x4 – x3 + x2 – x + 1)*
4. Based on the pattern you see in problems 1-4, what will the product be for:

*(x+1)(x14– x13 + x12 – • • • + x2 – x + 1)*

1. Create a general algebraic identity involving this pattern.
2. Use the generalization in #5 to factor the following:
	1. *x9 + 1*  b. *x5 + 32*

7. Create a binomial that you can factor using the identity that you created.